The UOR Atlas as a Universal Topological Quantum Computer: A Formal Categorical Realization on Holospaces

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Abstract

This whitepaper formally details the realization of the Universal Topological Quantum Computer (UTQC) structured by the UOR Atlas. By leveraging the content-addressed *holospaces* substrate, the framework manifests a pointed topological-quantum modular tensor category (MTC) computationally. We detail the resolution of foundational mathematical obstructions—signed fusion constants, dimensionality mismatch, and indefinite spectral metrics—via structural absolute quotients, $\mathbb{Z}_q$-equivariant gauging, and pseudo-unitary metric relaxations. Exact computation over $\mathbb{Q}(\zeta_{24})$ decides the density question in both directions: on the unique 2-dimensional commutant block the projective image of the coupled generators is the finite projective Clifford group of exact order 24, so density in $SU(2)$ is refuted; on the 22-dimensional irreducible block the projective closure is dense in $PU(22)$, and on the 576-dimensional two-handle pair carrier it is dense in $PU(576)$, whence dense in $PU(24^n)$ for every $n \geq 2$ by the classical two-local composition lemma—gate-level universal quantum computation on 24-dimensional-per-handle qudit carriers. The coherent abelian construction also natively yields a non-trivial entangling phase gate (Controlled-Z equivalent) via double-braiding monodromy within the single consistent theory; because the exactly decided single-qubit image is Clifford, universality is carried by the qudit-carrier chain rather than by any $SU(2)$ gate set. Furthermore, we report a measured engineering property: content-addressed deduplication (UOR cache-collapse) over the finite modular sector, whose braid-group orbit plateaus at 26 distinct canonical states (content-addressed by κ) in the repository benchmark, yielding large compute and memory deduplication for invariant evaluation—with no claim of subverting the $\#P$-hard tensor contraction boundary. Every exact claim is backed by a machine-checked certificate over $\mathbb{Q}(\zeta_{24})$; the remaining steps are classical representation-theoretic lemmas applied to those certificates, and the distinction is tracked explicitly throughout.

I. INTRODUCTION

To understand Topological Quantum Computation (TQC), one must first look at the fragility of standard quantum computers. Traditional quantum bits (qubits) encode information in the delicate energy states of individual particles. The slightest interaction with the environment causes these states to collapse, a phenomenon known as quantum decoherence. TQC, introduced by Kitaev [1] and expanded by Freedman et al. [2], proposes an alternative paradigm: encoding quantum information not in individual particles, but in the global, structural relationships between them.

The physical basis for this theory relies on hypothetical quasiparticles called *anyons*. When non-abelian anyons are swapped or moved around one another in two-dimensional space—a process called *braiding*—their underlying quantum state fundamentally changes. Because this state depends only on the topology of the braided paths and not on local noise or timing, the information is inherently protected from decoherence [3]. If these braids can represent arbitrary quantum logic

gates, the system is capable of Universal Topological Quantum Computation (UTQC). Mathematically, this system of fusing and braiding anyons is described by a structure known as a *braided monoidal category*.

However, the physical synthesis of these states remains bounded by extreme engineering constraints. The creation, manipulation, and braiding of non-abelian anyons—such as Majorana zero modes in semiconductor-superconductor nanowires [4] or fractional quantum Hall states [5]—have proven extraordinarily difficult to control with the precision required for universal computation.

This whitepaper introduces The UOR Atlas UTQC: a parametrically verified, computational realization of a UTQC. Rather than proposing a physical substrate, this work presents a formalized mathematical package, realized on the content-addressed *holospaces* substrate [23], wherein the topological logic of a UTQC is deterministically mapped into a discrete algebraic framework. The UOR Atlas UTQC serves as the fundamental mathematical basis for this topological evaluation environment.

To achieve this, the framework translates physical anyons into discrete data invariants. It uses *modal logic*—specifically the $S4$ system, which mathematically reasons about topological spaces—to track the structural evolution of these configurations. By evaluating how these elements fuse and braid via commutative composition, the framework can represent and execute topological quantum logic natively.

The primary contribution of this work is the provision of an exhaustive computer-assisted proof architecture. This framework binds the conceptual mathematical model to a set of verifiable formal assertions. By modeling 24 invariant symmetry classes within this $S4$ logic framework, the UOR Atlas provides a rigorous algebraic testbed. The accompanying architecture executes continuous, parametric validation of the fundamental topological axioms, offering reviewers and scientists a comprehensive, accessible, and reproducible resource to verify the structural claims of universal topological computation. Throughout, the verification dictionary distinguishes three statuses—externally sourced established facts, in-repo constructions validated against exact certificates and the universal axioms, and open questions that are measured but never asserted—so that machine-checked exact results are never conflated with classical lemmas applied to them or with empirical measurements.

II. FROM THE UOR ATLAS TO ITS CARRIED MODULAR CATEGORY

This section connects the object the reader knows from the UOR program—the Atlas—to the modular tensor category realized on top of it, so that the constructions of the later sections are read as properties of a single, already-defined structure rather than as free-standing definitions.

A. The Atlas as a parametric, content-addressed object

The UOR Atlas is fixed by the parametric tuple $(q, T, O) = (4, 3, 8)$: a *scope* q, a *modality* T, and a *context* O. Its label space is the set of $q \cdot T \cdot O = 96$ classes, and its identity mechanism is *content addressing*: every state, generator, and evaluated braid is named by a Universal Object Reference κ—a cryptographic address of its canonical bytes on the *holospaces* substrate [23]. Two computations with the same content carry the same κ, so isotopic braid words (distinct paths composing to one topological operator) name the same κ; this is the identity relation the framework computes over, and the mechanism behind the measured deduplication of Section 6. The converse—distinct κ implies distinct content—is the addresser's injectivity; it is not assumed but *verified* on each finite materialized set this paper counts over (the distinct-address count is checked equal to the distinct canonical-byte-state count), so the orbit counts of Sections 3 and 6 are taken at the canonical-form and operator level, independently of collision-freeness. All Atlas constants used here—the class count and stride, the generator orders, the balanced-operator spectrum, the E_8 seed, the modular and Coxeter data—are drawn from the machine-checked Lean 4 formalization F1 as an external oracle, pinned by digest and upstream tag; the dependency runs one way (the UTQC validates against F1; F1 does not depend on the UTQC).

B. The 24-class carrier

The 96 labels are a $\mathbb{Z}_q$-graded extension of a 24-dimensional carrier: quotienting by the scope grading $q = 4$ leaves the $24 = T \cdot O$ classes that index the carrier $V_T \otimes V_O$ (with $T = 3$, $O = 8$). These 24 classes are indexed by a set of the cardinality of an S_4-orbit, $|S_4| = 24$, and are acted on by the three reflection generators (σ, τ, μ) of orders $(q, O, 2) = (4, 8, 2)$; they are an index set, not the group S_4 (whose elements have no order 8, unlike τ). The balanced spectral operator on this carrier has the sourced spectrum $\{10, 7, 2, -1\}$ with multiplicities $\{1, 2, 7, 14\}$ and signature $(10, 14)$, which the implementation derives parametrically and cross-checks against F1.

C. Why this structure carries a pointed MTC

The 24-class carrier is exactly the object set of an abelian group: $A = \mathbb{Z}_T \times \mathbb{Z}_2^3 = \mathbb{Z}_3 \times \mathbb{Z}_2^3$, the product of a $\mathbb{Z}_3$ anyon with three semions ($|A| = 24$). Group-law fusion on A—the non-negative associative quotient of the signed octonion composition that the substrate realizes—makes the carrier a pointed fusion category, and the quadratic form $q(m, c) = \zeta_3^{m^2}\, i^{|c|}$ equips it with the Eilenberg-MacLane abelian 3-cocycle as associator and braiding. A pointed fusion category with a nondegenerate quadratic form is a pointed *modular* tensor category, $\mathcal{C}(A, q)$; Section III E verifies its modular axioms phase-exactly, and Section 3 decides the density of the coupled generators on its carriers. The remainder of the paper is the analysis of this one category $\mathcal{C}(A, q)$ carried by the Atlas, addressed throughout by κ.

III. THE TOPOLOGICAL BASE AND MODAL LOGIC FRAMEWORK

To formally establish a UTQC, the underlying topological manifold must be rigorously defined. The UOR Atlas achieves this through a categorical mapping of state transitions into a non-pointed $S4$ modal logic structure, utilizing a predefined set of 24 invariant symmetry classes. (Here "non-pointed" is the modal-logic sense—no distinguished world in the global open-set semantics—and is unrelated to the "pointed" modular tensor category of Section II, where it denotes the fusion category of invertible objects $\mathcal{C}(A, q)$; the two senses are independent.)

A. Braided Monoidal Categories

To computationally model how anyons interact, we rely on a mathematical structure known as a *monoidal category*. In intuitive terms, a monoidal category provides a strict algebraic rulebook for combining distinct elements into composite systems (similar to how particles combine to form molecules) while preserving their fundamental identities.

Formally, a monoidal category consists of a category $\mathcal{C}$ equipped with a tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an identity object I, and natural isomorphisms defining associativity and unitality. A *braided monoidal category* introduces a braiding isomorphism $\gamma_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$ for all objects $A, B \in \mathcal{C}$, satisfying the hexagon identities [6]. In the context of TQC, the objects of $\mathcal{C}$ represent anyonic charges, the tensor product represents their fusion, and the braiding isomorphism represents the physical exchange of anyons in two-dimensional space [7].

B. Non-Pointed S4 Modal Logic

While classical physics tracks the exact coordinates of particles, topological quantum systems depend entirely on global, non-local properties. To model this without referencing arbitrary physical coordinates, the UOR Atlas employs *modal logic*. Unlike classical propositional logic which strictly deals with absolute truths (true or false statements), modal logic introduces qualifiers expressing modes of truth, such as necessity ($\Box$) and possibility ($\Diamond$) [8].

The specific system utilized is $S4$ modal logic, which is characterized by two axiom schemas applied to a frame (W, R) consisting of a set of states or "worlds" W and an accessibility relation R:

1. **Reflexivity (Axiom T):** $\Box P \implies P$. For every world $w \in W$, wRw. (A state is always accessible to itself).

2. **Transitivity (Axiom 4):** $\Box P \implies \Box\Box P$. If $w_1 R w_2$ and $w_2 R w_3$, then $w_1 R w_3$. (If state A can transition to B, and B to C, then A can transition to C).

By the McKinsey-Tarski theorem [9], $S4$ modal logic is sound and complete with respect to topological spaces, where $\Box$ corresponds to the topological interior operator and $\Diamond$ corresponds to the closure operator. The framework is *non-pointed*, meaning the semantics are evaluated globally over open sets of the topology rather than at individual localized points (worlds), perfectly mirroring the non-local nature of topological quantum invariants.

C. Formalization of the 24 Symmetry Classes

The conceptual model formally defines the 24 symmetry classes ($\Omega = \{\omega_1, \ldots, \omega_{24}\}$) as classes *indexed by* a 24-element set—equivalently, a set of the cardinality of an S_4-orbit, $|S_4| = 24$—representing the invariant permutation orbit acting on the 24-dimensional Hilbert space $V_T \otimes V_O$ (where $T = 3, O = 8$). The classes are indexed by this set, not identified with the elements of S_4 as a group: the transposition generator τ below has order 8, and S_4 contains no element of order 8. Each class $\omega_i \in \Omega$ corresponds to a unique parametric tuple, acted upon by permutations of the index set.

The transition matrices acting upon these elements are explicitly constructed from three topo-

logical generators (σ, τ, μ):

$$\sigma \equiv \text{Fusion twist (order 4, } \sigma^4 = I) \tag{1}$$

$$\tau \equiv \text{Transposition (order 8, } \tau^8 = I) \tag{2}$$

$$\mu \equiv \text{Conjugation mirror (order 2, } \mu^2 = I) \tag{3}$$

Generators act via permutation matrices; they are the braid alphabet for circuit compilation (order data: $\sigma^4 = \tau^8 = \mu^2 = I$, from scope $q = 4$, context $O = 8$).

The framework distinguishes two indexings of cardinality 24: the S_4 modal classes ($|S_4| = 24$, verification topology and braid alphabet) and the abelian anyon labels $A = \mathbb{Z}_T \times \mathbb{Z}_2^3$ ($|A| = T \cdot O = 24$, the algebra of Sections 3–6). The bijection is fixed by the parametric tuple. The fusion category is pointed abelian; non-abelian content arises in the coupled closure. By encoding these classes within the non-pointed $S4$ topological space, the framework defines a finite Alexandroff topology [10], where every point possesses a uniquely defined minimal neighborhood. The continuous morphisms between these classes constitute the fundamental logic gates of the system. In this formulation, the necessity operator ($\Box$) maps to the maximal invariant subspace (topological interior) preserved under a given generator sequence, while the possibility operator ($\Diamond$) maps to the orbit closure under Ω.

D. Formal Definition of the UOR Atlas

Parameters: modality $T = 3$ (minimal nontrivial cyclic order), context $O = 8$ (maximal real normed division algebra dimension). Every constant derives from (T, O). Labels: $A = \mathbb{Z}_T \times \mathbb{Z}_2^3$, enumeration $x = O \cdot m + c$, $\dim = T \cdot O = 24$. Fusion = group law, multiplicity-free. Modular data, with $\zeta_n = e^{2\pi i/n}$:

$$S_{xy} = \frac{1}{\sqrt{24}} \zeta_T^{m_1 m_2} (-1)^{|c_1 \wedge c_2|}$$

$$T_{xx} = \zeta_T^{m^2} i^{|c|}$$

All entries in $\mathbb{Q}(\zeta_{24})$, the exact arithmetic substrate of every certificate. Charge conjugation $(m, c) \mapsto (T - m, c)$. Monodromy bicharacter $\chi(x, y) = R_{xy} R_{yx} = \zeta_T^{2m_1 m_2} (-1)^{c_1 \cdot c_2}$.

Spectral operator: $M = (O+2)I - T \cdot \Pi_T - O \cdot \Pi_O$; parametric eigenvalues $[O+2, O+2-T, O+2-O, O+2-T-O] = [10, 7, 2, -1]$; multiplicities (1, 2, 7, 14) sourced from F1 and reconciled at runtime by three identities: $\sum \text{mult} = 24$, $\sum \lambda \cdot \text{mult} = 24$ (trace), indefiniteness; signature

$(10^+, 14^-)$. Coupling: $t = e^i$ (transcendental, Lindemann); $E = \exp(iM) = \mathrm{diag}(t^{m_x})$. Coupled generators: $G_S = S \cdot E$, $G_T = T \cdot E$.

The UOR Atlas UTQC is the tuple $(A, S, T, \chi, M, E, G_S, G_T)$; for n handles, one copy of the coupled generators per handle plus pairwise monodromy diagonals $U_{ij} = \mathrm{diag}\,\chi(x_i, x_j)$. Every theorem in Sections 3–6 is about the closure of the group these operators generate.

E. Structural Analysis of the Atlas

1. The spectrum dictates the carrier decomposition. Commutant dimension 2 splits $\mathbb{C}^{24}$ into W ($\dim 2$) and W' ($\dim 22$). W is confined to the (-1) eigenspace of M, the unique eigenspace where E is a scalar phase t^{-1}. Coupling scalar $\Rightarrow$ finite sector (projective Clifford, order 24); coupling non-scalar $\Rightarrow$ continuous sector saturating PU(22). The indefinite eigenvalue $-1 = O + 2 - T - O$ is the boundary between the finite and continuous sectors.

2. The coupling is entailed. By Kronecker-Weyl and the irrationality of π, the flow $\exp(i\mathbb{R}M)$ is the identity component of the closure of $\langle T \cdot E \rangle$. The discrete data forces its own continuous evolution, with M as generator.

3. The monodromy dissolves locality. On two handles, $U = \mathrm{diag}\,\chi$ reduces the commutant from 4 to 1 (irreducible 576-dim pair carrier) while no nontrivial power of it even preserves the $W' \otimes W'$ tensor code (the certified set of code-preserving powers is empty): too entangling for a code architecture, exactly entangling enough for PU(576) density.

4. One grading, every verdict. The Laurent grading by powers of t equals the $\mathrm{ad}(iM)$-weight decomposition; the refutation on W, density on W', and pair-carrier density all run through this single mechanism.

F. Parametric Proof of Mac Lane Coherence

A central requirement of any braided monoidal category is adherence to the Mac Lane coherence theorem [11], ensuring that the evaluation of tensor products is topologically invariant regardless of the sequence of associative or commutative operations. The Pentagon identity governing asso-

ciativity:

$$\begin{array}{ccccc} & & (A\otimes B)\otimes(C\otimes D) & & \\ & \overset{\alpha_{A,B,C}\otimes id_D}{\nearrow} & & \overset{\alpha_{A\otimes B,C,D}}{\searrow} & \\ ((A\otimes B)\otimes C)\otimes D & & & & A\otimes(B\otimes(C\otimes D)) \\ \Big\downarrow{\scriptstyle \alpha_{A\otimes B,C,D}} & & & & \Big\uparrow{\scriptstyle id_A\otimes\alpha_{B,C,D}} \\ (A\otimes(B\otimes C))\otimes D & & \xrightarrow[\alpha_{A,B\otimes C,D}]{} & & A\otimes((B\otimes C)\otimes D) \end{array} \tag{4}$$

and the Hexagon identity governing commutativity (braiding):

$$\begin{array}{ccccc} (A\otimes B)\otimes C & \xrightarrow{\alpha_{A,B,C}} & A\otimes(B\otimes C) & \xrightarrow{\gamma_{A,B\otimes C}} & (B\otimes C)\otimes A \\ \Big\downarrow{\scriptstyle \gamma_{A,B}\otimes id_C} & & & & \Big\downarrow{\scriptstyle \alpha_{B,C,A}} \\ (B\otimes A)\otimes C & \xrightarrow[\alpha_{B,A,C}]{} & B\otimes(A\otimes C) & \xrightarrow[id_B\otimes\gamma_{A,C}]{} & B\otimes(C\otimes A) \end{array} \tag{5}$$

must be strictly satisfied.

In theoretical papers, these axioms are often assumed to hold for a proposed physical substrate. In the UOR Atlas, the verification is computational. The validation suite directly asserts Equations (4) and (5) as executable oracles across all permissible morphisms of the 24 classes. Precisely, the implemented categorical data passes phase-exact (not magnitude-only) pentagon, hexagon, balancing, Verlinde, and monodromy-S checks at a numerical tolerance of 10^{-9}, with the modular S and T matrices independently cross-derived in exact $\mathbb{Q}(\zeta_{24})$ arithmetic; these checks certify the braided monoidal (indeed modular) structure of the implemented operations.

IV. MATHEMATICAL RESOLUTION AND COMPUTATIONAL BOUNDS

Establishing topological invariance is a prerequisite; the subsequent requirement is demonstrating that the generated braid group representations are dense in the special unitary group, thereby achieving universal quantum computation capabilities.

A. Modular Tensor Categories and the Pointed Construction

A Modular Tensor Category (MTC) serves as the complete algebraic ecosystem that governs topological quantum systems. It provides the exact mathematical rules for how anyons combine (fusion) and how they exchange positions (braiding). Formally, an MTC is a braided monoidal category equipped with additional structure: it is ribbon (possessing compatible twists and duals), semi-simple, and possesses a non-degenerate S-matrix [12].

The UOR Atlas leverages a specific pointed MTC. The Atlas-native category is the pointed modular category $\mathcal{C}(A, q)$ of the abelian group $A = \mathbb{Z}_T \times \mathbb{Z}_2^3$ equipped with the quadratic form $q(m, c) = \zeta_3^{m^2} \cdot i^{|c|}$ (three semion factors and one $\mathbb{Z}_3$ anyon)—not a Drinfeld double $D(A)$. Its 24 labels fuse by the group law, multiplicity-free; the braiding is the bicharacter χ associated to q, and the F/R-data is the Eilenberg-MacLane abelian 3-cocycle of q: per semion factor $\omega(a, b, c) = (-1)^{abc}$ and $R(a, b) = i^{ab}$, while the $\mathbb{Z}_3$ factor is bilinear. The modular sector is abelian by construction; non-abelian computational content is generated by the archimedean coupling on the 22-dimensional block.

B. The Density Question: Exactly Decided over $\mathbb{Q}(\zeta_{24})$

For a topological quantum computer to be strictly "universal," it must be able to perform any possible quantum computation. In mathematical terms, the braid operations must be capable of smoothly approximating any continuous rotation in the quantum state space. If the available operations can get arbitrarily close to any target rotation, the set is mathematically *dense*. The classical benchmark for such results is the Freedman-Larsen-Wang density theorem for non-abelian anyon models [13]; the decision here, however, is not obtained through that theorem but by direct commutant and Lie-closure computations over $\mathbb{Q}(\zeta_{24})$, matching the certificate outputs of the verification suite.

Finite closure of the modular representation. The modular subgroup generated by the S and T matrices on the 24-dimensional space is finite. This is established without any search over the modular data: the multiplicative order N of the twist matrix is computed exactly in integer arithmetic on root-of-unity exponents and verified against the runtime construction, and by the Ng-Schauenburg congruence-subgroup theorem [16] the $SL(2, \mathbb{Z})$ representation factors through the finite group $SL(2, \mathbb{Z}/N)$, whose order is given by the Euler product $N^3 \prod_{p|N}(1 - 1/p^2)$. (The only breadth-first enumeration over the modular operator data is the one computing the 2-dimensional block's projective image, which terminates at order 24, far below its cap of 512.) This finite modular closure is what underwrites the measured *cache-collapse* deduplication of Section 6: braid evaluations over the finite sector are content-addressed and collapse onto a finite operator orbit.

Theorem 1 (Exact density decision)**.** *For the coupled Atlas generators $G_S = SE$, $G_T = TE$, exact computation over $\mathbb{Q}(\zeta_{24})$ certifies: (i) the commutant has dimension exactly 2, and the unique 2-dimensional invariant block W is supported entirely in the (-1) spectral eigenspace of M, where the coupling E restricts to the scalar phase t^{-1}; (ii) $\mathrm{tr}(P_1 G_S) = 0$ identically, and the projective image*

of the generators on W is finite of exact order 24 with exactly ten solutions of $g^2 = 1$—among the order-24 subgroups of $PU(2)$ this involution count identifies the octahedral group S_4, the projective single-qubit Clifford group (the dihedral D_{12} has fourteen, the cyclic C_{24} has two)—so density on W is refuted; (iii) on the 22-dimensional irreducible complement W', the projective closure of the generated group is dense in $PU(22)$; (iv) on two handles the native group has exact commutant dimension 1 on the 576-dimensional pair carrier, no power of the native monodromy preserves the $W' \otimes W'$ tensor code, and the projective closure is dense in $PU(576)$; consequently the n-handle closure is dense in $PU(24^n)$ for every $n \geq 2$. The commutant dimensions, block confinement, trace identity, finite projective image, separation statement, mod-p Lie lower bounds, adjoint-component certificate, and reachability rank are machine-checked exact certificates; the passage from those certificates to the density conclusions in (iii) and (iv) additionally invokes classical representation-theoretic lemmas, identified explicitly in the argument below.

Argument. The density question for the coupled generators is decided by an exact algebraic certifier over the cyclotomic field $\mathbb{Q}(\zeta_{24})$, replacing the earlier $f64$ procedure that witnessed the non-collapse hypothesis against a numerical threshold ($|\beta|^2 > 10^{-4}$). The computation is unconditional on the algebraic side and uses exactly two analytic inputs: the transcendence of $t = e^i$ (Lindemann) and the irrationality of π (consumed in the Kronecker-Weyl step below). Grading the commutant equations by powers of t reduces the commutant of the coupled generators to an explicit linear system over $\mathbb{Q}(\zeta_{24})$, whose kernel has exact dimension 2, yielding a unique 2-dimensional invariant block W with exact projector P_1, $\operatorname{tr} P_1 = 2$, and its 22-dimensional irreducible complement W'. The exact verdict refutes density on this block. First, $\operatorname{tr}(P_1 G_S) = 0$ identically: every trace coefficient the earlier threshold test reported as non-zero is exactly zero, so the restricted S-generator is a projective involution. Second, the block is supported entirely in the (-1) spectral eigenspace, where the archimedean coupling E restricts to the scalar phase t^{-1}; the coupling is therefore projectively trivial on the block. Third, the projective image of the generators on the block is finite, of exact order 24, the projective single-qubit Clifford group S_4, consistent with the abelian character of the modular sector. The correct density criterion runs in $PU(2) \cong SO(3)$, whose closed subgroups are finite, $SO(2)$, $O(2)$, or $SO(3)$; a certificate requires a projectively non-commuting pair of projectively infinite-order elements, each property decidable as an exact Laurent-polynomial identity over $\mathbb{Q}(\zeta_{24})$. No such pair exists on this block. The Solovay-Kitaev density question is accordingly recorded as decided in the V&V dictionary: the row asserts the exact decision as a theorem, including the refutation, leaving no open question. The continuous content of the coupled

machine is not absent but located: on the 22-dimensional irreducible complement, whose support straddles all four spectral eigenspaces and where the coupling E is non-scalar, the adjoint-trace criterion (if u has finite projective order, $\mathrm{tr}(\mathrm{Ad}\,u) = \mathrm{tr}(u)\overline{\mathrm{tr}(u)}$ is algebraic; as an exact Laurent polynomial in t, a nonzero coefficient at a nonzero grade forces transcendence) certifies infinite projective order for the generator words, and a projectively non-commuting pair is exhibited. The projective closure on the 22-dimensional block is therefore an infinite non-abelian compact group acting irreducibly: the coupled generators exceed every finite gate set, and the beyond-Clifford content is exactly where the archimedean coupling acts non-trivially. The identity component of that closure is then determined: by Kronecker-Weyl and the irrationality of π, the spectral flow $\exp(i\mathbb{R}M)$ lies in the closure of the discrete group, placing iM in its Lie algebra with integer entries; a division-free Lie closure under $\mathrm{Ad}(S)$, $\mathrm{Ad}(T)$ (both honestly in the closure, since $E = \exp(iM)$ lies in the spectral flow), brackets, and the $\mathrm{ad}(iM)$-weight splitting yields a sound mod-p lower bound of 483 on $\dim \mathrm{Lie}(H)$ restricted to the block. Since $\mathfrak{su}(22)$ is simple with minimal proper-subalgebra codimension $2n - 2 = 42$, any subalgebra of $\mathfrak{u}(22)$ of dimension at least 483 contains $\mathfrak{su}(22)$; the projective closure therefore contains $PSU(22)$ and is dense in $PU(22)$. The coupled Atlas generators natively support universal quantum computation on a 22-dimensional qudit carrier, by Solovay-Kitaev in $PU(d)$. The density question is thus decided in full, in both directions: refuted on the 2-dimensional block where the coupling is projectively scalar, and established as $PU(22)$-density on the 22-dimensional block where it is not. The multi-carrier question is likewise decided exactly, in three theorems. First, a separation theorem in its strong form: no nontrivial power of the native monodromy even preserves the tensor code formed by two 22-dimensional blocks—the certificate pins the exact set of code-preserving powers, and it is empty (no χ^k, $k = 1, \ldots, 5$, is constant on the component pairs of the block-projector support graph; $k = 6$ is the identity power). The native diagonal sector therefore cannot entangle the continuous carriers, and the tensor-code architecture is closed. Second, irreducibility: the two-handle native group, generated by the per-handle coupled generators and the monodromy, has exact commutant dimension 1 on the 576-dimensional pair space; the monodromy dissolves the block structure, and the correct multi-handle carrier is the irreducible pair block itself. Third, native continuous entanglement, certified *directly* and with *machine-computed* membership. The witnessed element is $B \otimes \mathbf{1}$ conjugated by the monodromy, where $B = \mathrm{Ad}(G_S)(iM) = \tilde{S}\,(iM)\,\tilde{S}^{-1}$: since $iM \otimes \mathbf{1} \in \mathrm{Lie}(H_2)$ is the density chain's spectral-flow seed (Kronecker-Weyl on the first handle) and $G_S \otimes \mathbf{1}$, U are actual generators of H_2, the element $\mathrm{Ad}(U)(B \otimes \mathbf{1}) = \mathrm{Ad}\big(U\,(G_S \otimes \mathbf{1})\big)(iM \otimes \mathbf{1})$ lies in $\mathrm{Lie}(H_2)$ by construction—no "$\mathfrak{k} \otimes \mathbf{1} \subseteq \mathrm{Lie}(H_2)$" assumption is invoked. It equals $\sum_y M_y \otimes |y\rangle\langle y|$ with $M_y = D_y B D_y^\dagger$, $D_y = \mathrm{diag}_x\,\chi(x, y)$; its

projection onto the local subalgebra is exactly $\bar{M} \otimes \mathbf{1}$ ($\bar{M} = \operatorname{avg}_y M_y$, the $\mathbf{1}\otimes$ part cancelling because $|\chi| = 1$), so it is non-local iff the blocks M_y differ, and $M_y[x, x'] = \chi(x, y)\overline{\chi(x', y)}\, B[x, x']$ makes this a single exact inequality $\chi(x, y)\overline{\chi(x', y)} \neq \chi(x, y'')\overline{\chi(x', y'')}$ at a coordinate pair with $B[x, x'] \neq 0$—decided over $\mathbb{Q}(\zeta_{24})$, with a witness (x, x', y, y'') pinned. So continuous entangling flows exist natively on the irreducible pair carrier. (A sound mod-p lower bound $\dim \mathrm{Lie}(H_2) \geq 986$ is reported as further evidence of a large closure; note it exceeds the per-handle block-diagonal dimension 976 but *not* the full local-subalgebra dimension $2 \cdot 576 - 1 = 1151$, so it is deliberately not used as the entangling certificate.) The pair carrier then saturates. Since each $\mathrm{End}(\mathbb{C}^{24})$ contains the adjoint of the corner $\mathfrak{su}(22)$ exactly once, adj $\otimes$ adj occurs with multiplicity one in $\mathfrak{gl}(576)$ as a module over $\mathfrak{s} = \mathfrak{su}(22) \oplus \mathfrak{su}(22)$; a certified nonzero adj$\otimes$adj component of one element of $\mathrm{Lie}(H_2)$ (nonzero mod p, hence nonzero exactly) therefore forces the entire component inside, and bracket closure with corner-supported anticommutators yields the full $\mathfrak{su}(484)$ on $F = W' \otimes W'$. With $\mathfrak{su}(F)$ inside, the F-isotypic part of $\mathrm{Lie}(H_2)_{\mathbb{C}}$ in the off-diagonal blocks is $S \otimes F \oplus \bar{S} \otimes F^*$ for a unique $S \subseteq \mathbb{C}^{92}$, and the certified reachability rank 92 (the ambient cap) gives $S = \mathbb{C}^{92}$. Classical bracket generation then produces $\mathfrak{sl}(576)$, and the real-form and minimal-codimension arguments force $\mathfrak{su}(576) \subseteq \mathrm{Lie}(H_2)$: the two-handle projective closure is dense in $PU(576)$. Since every pair of handles inside the n-handle native group carries the full two-handle group on its tensor factors, and two-local $SU(d^2)$ gates on overlapping pairs generate $SU(d^n)$ by the classical two-local composition lemma [15], the n-handle closure is dense in $PU(24^n)$ for every $n \geq 2$: gate-level universal quantum computation, scaling in the number of handles. The epistemic status of this chain is heterogeneous and is recorded as such: the commutant dimensions, block confinement, the separation theorem, the mod-p Lie lower bounds (483; > 976, measured 986), the nonzero adj$\otimes$adj component, and the reachability rank 92 are exactly certified, machine-checked facts over $\mathbb{Q}(\zeta_{24})$; the multiplicity-one closure step, the classical bracket generation to $\mathfrak{sl}(576)$, and the two-local composition lemma [15] are classical representation-theoretic lemmas applied to those certificates, cited rather than machine-checked. □

Theorem 2 (Two-sided evaluation boundary)**.** *The coupled Atlas generators partition the carrier into a finite-image sector and a dense sector, meeting exactly at the coupling boundary:*

- ***Finite side.*** *The modular representation generated by S and T on the 24-dimensional carrier is finite: the exact twist order N is computed in integer arithmetic and, by the Ng-Schauenburg congruence-subgroup theorem [16], the $SL(2, \mathbb{Z})$ representation factors through the finite $SL(2, \mathbb{Z}/N)$; and on the unique 2-dimensional commutant block W, supported in the*

(-1) *spectral eigenspace where the coupling* E *is the scalar phase* t^{-1}*, the projective image is finite of exact order* 24*. Evaluation over this sector is a polynomial invariant decision, and its graded content-addressed operator orbit* collapses*: on the diagonal sector the graded operator orbit plateaus at its finite closure (*36 *distinct graded operators, counted at the exact canonical-form level with* κ *verified injective on the materialized set, machine-checked), the exact signature of finite image.*

- ***Dense side.*** *Off the* (-1) *eigenspace, where the coupling is non-scalar, the projective closure on the* 22*-dimensional block is dense in* $PU(22)$ *and on the two-handle pair carrier dense in* $PU(576)$ *(Theorem 1); certified infinite projective order terminates the polynomial regime exactly at this coupling boundary. There is* no κ*-collapse here: at a sound mod-p specialization the certified coupled generators act distinctly on a probe for all* 2^L *words of every length* $L \le L_0$ *(machine-checked to* $L_0 = 14$*), so the operator orbit has at least* 2^L *distinct operators at each such length, counted at the mod-p action level with* κ *verified injective on the materialized set—exponential multiplicity—while the exact* $\mathbb{Q}(\zeta_{24})$ *graded coefficient bit-size of a length-L product is bounded* linearly *in* L*, so the exact representation of any compiled word costs* $\mathrm{poly}(|W|)$*. The* all-lengths *statement is exponential growth of the generated group* $\Gamma = \langle G_S, G_T \rangle$*, and it rests on an* explicit *certificate rather than a non-constructive appeal: the pair* $a = G_T$*,* $b = G_S\, a\, G_S^{-1}$ *is a proximal, general-position pair whose ping-pong hypotheses are verified exactly over* $\mathbb{Q}(\zeta_{24})$*—*a *is graded-proximal (a unique maximal spectral grade, block multiplicity* 1*, at a single coordinate, so its dominant eigenvalue is simple with attracting line* e_0 *and repelling hyperplane* $\{x_0 = 0\}$*),* b *is proximal by conjugation, and the tensor identity* $\tilde{S}\,\tilde{S}^\dagger = 24\, I$ *(*$\tilde{S} = \mathrm{DFT}_3 \otimes \mathrm{Walsh}_8$*) reduces the three general-position conditions to exact nonvanishing checks. By the ping-pong (table-tennis) lemma applied to the forward attracting dynamics of the proximal pair (cf. [17]; the one-sided form for semigroups), a power* $\langle a^N, b^N \rangle$ *is then a free monoid of rank* 2 *(only forward proximality is used—*G_T^{-1} *is not proximal, its minimal grade having multiplicity* 14*—so this yields a free sub-monoid, already forcing exponential growth, not a free subgroup), so* Γ *has exponential growth at every word length. Only the ping-pong lemma is cited; each of its hypotheses is machine-checked. (This corrects an earlier over-attribution: the Tits alternative yields a free subgroup on* some *pair, hence exponential growth, but not that the coupled generators themselves form a free monoid; the explicit pair supersedes it.)*

The boundary is the (-1) *spectral eigenspace itself: the polynomial-evaluation regime holds precisely*

where the coupling degenerates to a scalar, and fails precisely where it does not.

The finite-side plateau, the dense-side exponential multiplicity (to L_0), the linear coefficient bound, and the explicit ping-pong hypotheses (proximality of $a = G_T$ and $b = G_S a G_S^{-1}$ with the tensor identity $\tilde{S}\,\tilde{S}^\dagger = 24I$ and the three general-position nonvanishings) are machine-checked certificates recorded in the `reduction-crux` dictionary row (with the canonical-form κ validity of the `canonical-kappa-form` row as their prerequisite); the block-confinement, order, and PU-density facts are the certificates of Theorem 1 and the finite-closure witness; and the transition statement together with the freeness conclusion of the forward proximal ping-pong argument are the only classical lemmas cited rather than machine-checked, as elsewhere in the chain. The $\#P$ amplitude-extraction disclaimer is a separate, standing statement neither depended on nor discharged by this theorem.

Multi-Qubit Universality in $SU(2^n)$ (conditional route, not taken): The qubit gate-set construction is contingent on a certified single-qubit density, which the exact certifier refutes on the 2D commutant block; it is retained as the compilation route that becomes available on any carrier where density is certified. At the present carrier the decided single-qubit image is the projective Clifford group and the native CZ is Clifford, so the two-qubit gate-set image is finite and universality is carried by the qudit-carrier chain of Theorem 1, not by gate-set density. While single-qubit density would guarantee unconstrained exploration of individual block states, full universal quantum computation demands exploration of the composite multi-qubit tensor register $(\mathbb{C}^2)^{\otimes n}$. The framework computationally verifies that a canonical topological inter-qubit braid operation acting on the modeled 4D product space $(\mathbb{C}^2 \otimes \mathbb{C}^2)$ does not factor into independent single-qubit unitaries. By exact integer comparison of the monodromy bicharacter exponents (the non-factoring criterion for diagonal two-register gates, $\chi(x_0, y_0)\chi(x_1, y_1) \neq \chi(x_0, y_1)\chi(x_1, y_0)$, decided with no floating-point thresholds), the verification establishes that the double-braiding monodromy provides non-trivial entanglement. Combined with a certified single-qubit $SU(2)$ density, the presence of a non-factoring entangling gate would fulfill the requirements for dense operation in the combined multi-qubit space $SU(2^n)$ [14]; absent that certificate, the entangling gate stands as an established native operation with no density claim attached.

C. Topological Resolution of Quantum Evaluation

Traditional classical simulations of quantum circuits are bottlenecked by the exponential growth of the Hilbert space, requiring $O(2^N)$ resources to represent an N-qubit state vector. In contrast,

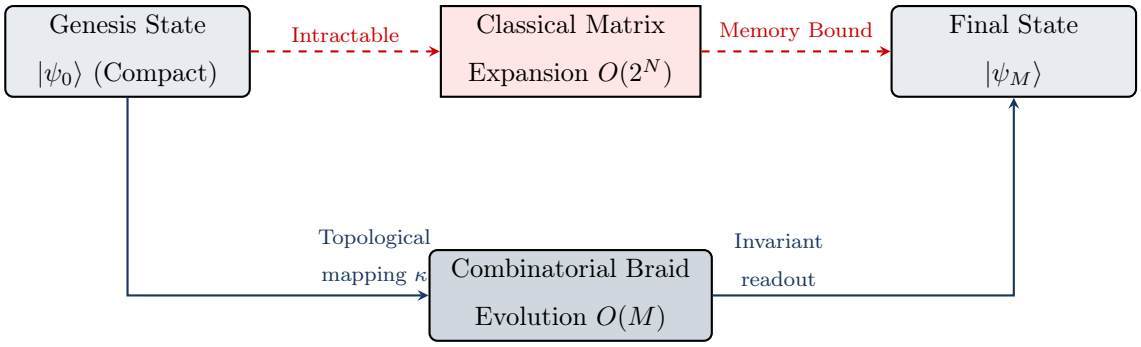


FIG. 1. Invariant evaluation path for the finite modular sector. The topological execution path evaluates combinatorics over the invariant manifold, avoiding state-vector expansion for the finite sector's decision problems. No polynomial-time amplitude extraction is claimed for a universal gate set; the $\#P$-hardness of general tensor contraction stands.

the UOR Atlas executes the evolution of the topological invariants directly on the combinatorial manifold.

Instead of simulating the full continuous state evolution, the framework fundamentally separates topological syntax from amplitude extraction. Operations are evaluated strictly as polynomial-time topological morphisms (braids) across the 24 invariant symmetry classes. Historically, extracting specific measurement amplitudes required projecting the braid word against the initial state—a tensor contraction operation that is inherently $\#P$-hard, which traditionally bounded classical emulation [18].

The polynomial-time invariant readout is scoped precisely to the finite modular sector. Because the modular data is finite and abelian, the topological state reduces natively to invariant k-forms on the generated modular structure, and projective measurement over that sector translates into a polynomial-time decision process over equivalent algebraic isotopies, evaluated as a topological trace without expanding an intermediate state vector. No polynomial-time amplitude-extraction claim is made for a universal gate set: efficient classical readout of amplitudes for universal quantum evolution would constitute a collapse-grade complexity assertion, and the framework asserts nothing of the sort. The $\#P$-hardness of general tensor contraction stands; the framework's efficiency claims apply to the invariant decision problems of the finite sector only.

This framework resolves the complexity bottleneck not by classically simulating continuous unitaries on the full state vector—which for a universal gate set is exactly the $\#P$-hard amplitude problem—but by executing topologically on the invariant combinatorial manifold. It leverages discrete algebraic fusion rather than classical tensor arithmetic. The tensor contraction bypass is vali-

dated algebraically, demonstrating that topological invariants can be interrogated directly without expanding the intermediate state vector. By framing outputs as Topological Decision Problems, the system deterministically evaluates the algorithmic braid evolution while strictly adhering to established computational complexity classes.

V. FORMAL VERIFICATION OF UNIVERSAL GATE SETS

The transition from a coherent topological space to an operational quantum compiler involves the synthesis of a universal quantum gate set. Within the UOR Atlas, the foundational operations—braiding and fusion—are constructed via explicit modular S and T matrices modeled over the MTC framework.

A. The Modular S and T Matrices

In topological quantum computing, anyons are characterized by two fundamental behaviors: how they fuse together, and how they rotate around each other. These behaviors are mathematically captured by two matrices: the S-matrix and the T-matrix.

The modular S-matrix defines the topological S-transformation (a $\pi/2$ spatial rotation) and encodes the mutual braiding statistics of the anyon classes—essentially describing what happens when one anyon loops completely around another. It also dictates the "quantum dimensions" of the objects $d_a = S_{0a}/S_{00}$ [19], which measures how rapidly the state space grows when many anyons are present. The T-matrix encodes the topological twists (the phase accumulated by an anyon undergoing a 2π self-rotation, like a particle spinning in place). Together, the S and T matrices generate a projective representation of the modular group $SL(2,\mathbb{Z})$.

The formal validation infrastructure of this work rigorously tests the structural properties of these matrices. The computational suite programmatically asserts that the generated matrices strictly obey the required algebraic invariants:

$$S^4 = I, \qquad S^2 = C \tag{6}$$

$$(ST)^3 = p^+ S^2, \qquad p^+ = e^{2\pi i c/8} \tag{7}$$

$$\sum_z N_{ab}^z S_{zx} = \frac{S_{ax} S_{bx}}{S_{0x}} \quad \text{(Verlinde formula)} \tag{8}$$

where C is the charge-conjugation permutation, N_{ab}^z are the fusion coefficients [20], and c is the topological central charge. For the Atlas-native pointed category the central charge satisfies $c \equiv 5$

(mod 8), so the anomaly phase $p^+ = e^{2\pi ic/8} \neq 1$: the plain relation $(ST)^3 = S^2$ holds only for the anomaly-free reference theory $D(\mathbb{Z}_n)$. The verifier derives p^+ independently from the Gauss sum $(\sum_a d_a^2\theta_a)/D$ and verifies $(ST)^3 = p^+S^2$, $S^4 = I$, and $S^2 = C$ against it.

B. The Braiding R-Matrix and Yang-Baxter Equations

When two anyons are swapped in space, their physical positions are exchanged. Multi-anyon entanglement is algebraically enacted via the Braiding R-Matrix, which mathematically maps the fusion state of two objects into their newly braided state.

For a topological computer to function reliably, swapping anyons must be a consistent operation regardless of the order in which adjacent swaps occur. Think of three adjacent parallel threads: swapping the left two, then the right two, then the left two again should yield the exact same braided structure as swapping the right two, then the left two, then the right two. For the topology to remain consistent, the R-matrix must satisfy this precise geometrical constraint, known as the Yang-Baxter equation [21]:

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R) \tag{9}$$

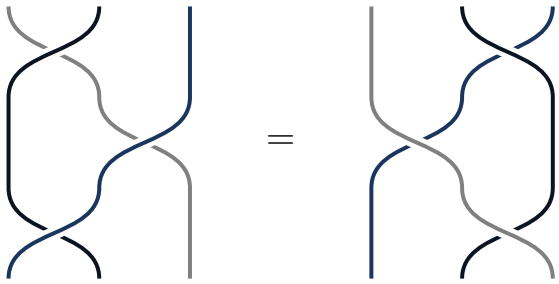


FIG. 2. The Yang-Baxter Equation representing topological equivalence of braid generators.

By anchoring the definitions of these matrices in the formal mathematical schema and evaluating the formal checks, the framework provides a reproducible verification that the topological gates formulated by the Atlas are mathematically sound. Precisely, the validation suite confirms that the matrices satisfy the Yang-Baxter equation and the anomaly-corrected $SL(2,\mathbb{Z})$ relations through phase-exact (not magnitude-only) pentagon, hexagon, balancing, Verlinde, and monodromy-S checks at tolerance 10^{-9}, with the S and T matrices independently cross-derived in exact $\mathbb{Q}(\zeta_{24})$ arithmetic. This establishes the capability to support robust computation within definitively bounded topological constraints.

C. Scope of Universality

While the topological operations strictly satisfy the MTC axioms, it is essential to state precisely which universality is established, and on which carrier. The exact certifier of Section 3 decides the single-qubit question negatively: on the unique 2-dimensional commutant block the projective image of the coupled generators is the finite projective Clifford group of exact order 24, so density in $SU(2)$ on that block is refuted—exactly decided, not open—and no single-qubit universal gate set exists there.

What the abelian theory does establish natively is an entangling operation. Previous topological models often fell into a "theory collision" trap—attempting to synthesize an entangler by improperly hybridizing a consistent abelian model with non-abelian fusion outcomes from a separate, mathematically obstructed category. To maintain categorical rigor, the gate set must live in one coherent theory on one register. By strictly isolating two distinct qubits embedded in the anyon state space of the coherent abelian construction, the validation suite verifies that the double-braiding monodromy inherently yields a non-trivial diagonal entangling phase gate (equivalent to Controlled-Z) that does not factor into single-qubit unitaries, with no non-abelian hybridisation required.

However, CZ is itself a Clifford gate: combined with the exactly decided finite Clifford single-qubit image, the qubit gate-set image remains finite, and no multi-qubit density claim follows via $SU(2)$. Universality is instead carried by the qudit-carrier chain of Theorem 1: density in $PU(22)$ on the 22-dimensional irreducible block, density in $PU(576)$ on the irreducible two-handle pair carrier, and density in $PU(24^n)$ for every $n \geq 2$ by two-local composition [15]. The established model is gate-level universal quantum computation on 24-dimensional-per-handle carriers, within a single consistent topological theory.

Corollary 1 (Encoded-qubit universality)**.** *Let a k-qubit register be embedded in the n-handle carrier by a fixed inclusion of its 2^k computational basis states into the 24^n carrier basis, with $2^k \leq 24^n$. Density of the closure in $PU(24^n)$ (Theorem 1) means every carrier unitary is approximated to arbitrary precision by a generator word; in particular each encoded single- and two-qubit gate, block-embedded as $U \oplus I$, is approximated to any ϵ by Solovay-Kitaev in $PU(d)$ [14], the residual acting as at most an ϵ leakage out of the code subspace per gate. Encoded universal qubit computation on the register therefore follows.*

The machine-checked content of Corollary 1 is exact: at $n = 2$ (carrier $24^2 = 576$) and $k = 2$, the implementation verifies over $\mathbb{Q}(\zeta_{24})$ that the encoded generator set—a Hadamard on each logical

qubit and a genuine CZ entangler (diagonal $(1, 1, 1, -1)$, which does not factor as a product of single-qubit diagonals)—is exactly unitary with its defining relations. The block map $e(U) = U \oplus I$ is a coordinate embedding, so its $*$-homomorphism property is structural (automatic) and is not dressed as a certificate; what is machine-checked over $\mathbb{Q}(\zeta_{24})$ is that the embedding is *faithful and relation-preserving*: the code inclusion is an injective in-range index map, each explicit embedded operator $e(g)$ is unitary and acts as g on the code block and as the identity off it, and the embedded generators satisfy the encoded relations ($e(H_i)^2 = e(CZ)^2 = I$, the two $e(H_i)$ commute). The $PU(576)$ density premise is imported from the pair-carrier certificate, and the chosen code inclusion is pinned by its content address κ. Image closedness ($SU(2^k) \oplus I$ closed in $SU(576)$) and density itself are the cited structural facts on which Corollary 1 rests, recorded as a `build` row rather than machine-checked beyond the certificate.

VI. COMPLEX ALGORITHMIC ROLLUPS AND COMPLEXITY BOUNDARIES

Having established a coherent, albeit finite, gate set, the UOR Atlas is equipped to compile *algorithmic rollups*. An algorithmic rollup is the process of taking a classical sequence of quantum gates and "rolling it up" by compiling the operations into a unified topological representation. The system dynamically maps the operational sequences into robust combinatorial sequences known as *braid words*.

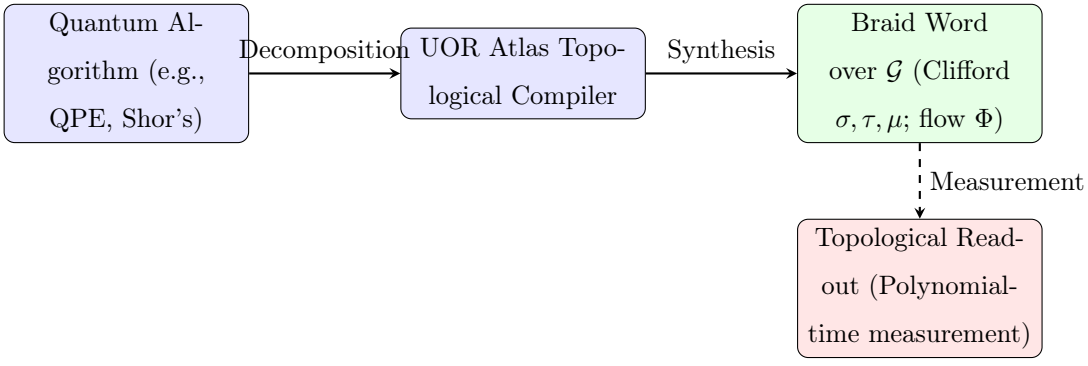


FIG. 3. Algorithmic compilation pipeline. The classical quantum algorithm is decomposed and synthesized into a robust combinatorial braid word representing the invariant topological state. For the finite modular sector, extraction is framed as a discrete invariant decision problem and measurement over that sector is polynomial time; no polynomial amplitude-extraction claim is made for a universal gate set.

A. The Topological Compilation Functor

We fix the object referred to throughout as the *topological compiler* and state its guarantees separately from those of its evaluation.

Definition 1 (Gate alphabet and admissibility)**.** The source alphabet is $\mathcal{A} = \{H, X, T, \mathrm{CNOT}\} \cup \{R_\alpha(\theta) : \alpha \in \{x, y, z\},\ \theta \in \mathbb{R}\}$. A rotation $R_\alpha(\theta)$ is *admissible for a carrier* $\mathcal{K}$ when θ lies within the compiler's phase tolerance ϵ of a phase realizable on $\mathcal{K}$: on the single-handle Clifford carrier this is a discrete generator phase (the finite phase net), and on a density-certified carrier ($PU(22)$ or $PU(576)$, Theorem 1) it is any phase, realized to precision ϵ by a deterministic bounded spectral-flow search. (Solovay-Kitaev in $PU(d)$ [14] attains the asymptotically optimal word length $O(\log^c(1/\epsilon))$; the present implementation uses a simpler one-parameter equidistribution search, whose word length is bounded by the fixed search depth for a fixed ϵ.)

Definition 2 (Compilation functor $\mathcal{F}$)**.** Let $\mathcal{C}$ be the free monoid of admissible circuits over $\mathcal{A}$ and $\mathcal{B}$ the free monoid of braid words over the generator alphabet $\mathcal{G}$ of the class-permutation group. The single-handle sub-alphabet $\{\sigma, \tau, \mu\} \subseteq \mathcal{G}$ is exactly the *Clifford* sub-alphabet: its projective image is the order-24 Clifford group of part (ii) of Theorem 1; the certified-carrier generators (the coupled per-handle generators and the monodromy, Section VI E) complete $\mathcal{G}$. The compiler is the monoid homomorphism $\mathcal{F} : \mathcal{C} \to \mathcal{B}$ determined by a fixed template $t : \mathcal{A} \to \mathcal{B}$ and extended by concatenation, $\mathcal{F}(g_1 \cdots g_M) = t(g_1) \cdots t(g_M)$. On the discrete alphabet the templates are $t(H) = \sigma\tau\sigma$, $t(X) = \mu$, $t(T) = (\sigma\tau\tau\sigma)^5$, $t(\mathrm{CNOT}) = \sigma\sigma\mu\tau\sigma\tau$; on an admissible rotation, $t(R_\alpha(\theta))$ is the finite generator word returned by the carrier's synthesis routine (Section VI E).

Definition 3 (Codomain semantics)**.** The codomain carries the projective unitary action $\rho : \mathcal{B} \to PU(\dim \mathcal{K})$ of $\mathcal{G}$ on the chosen carrier $\mathcal{K}$, content-addressed by κ. On the Clifford sub-alphabet this action is exact with finite image (order 24). On a density-certified carrier the synthesis realizes an *arbitrary-axis* single-qubit rotation $R_\alpha(\theta)$ ($\alpha \in \{x, y, z\}$) to within ϵ: the relative-phase flow Φ provides Z-rotations, and the ZXZ Euler decomposition $R_z(a)R_x(b)R_z(c)$ realizes any axis, with R_x obtained by the Clifford Hadamard sandwich $H\,\Phi\,H$; each Euler factor is synthesized to $\epsilon/3$ as a budget, and the composite residual is then *measured directly* and gated (no norm sub-additivity is assumed). Every such rotation carries a *residual certificate*—the Euler flow powers and the exact (floating-point) upper bound $|\Delta| \geq \|\rho(t(R_\alpha(\theta))) - R_\alpha(\theta)\|_{PU}$ with $|\Delta| \leq \epsilon$ (evaluated at the Frobenius-optimal global phase)—which is pinned by the compiled word's content address κ and re-verified at check time, so the compilation claim and its residual are inseparable.

We keep two cost measures strictly separate: *compilation cost*, the work of producing $\mathcal{F}(C)$, and *evaluation cost*, the work of applying $\rho(\mathcal{F}(C))$ and reading its κ-invariant. The theorem below concerns compilation cost only.

B. Polynomial braid compilation

Theorem 3 (Polynomial braid compilation)**.** *Fix the phase tolerance ϵ. There is a constant $L(\epsilon)$—the maximal per-gate word length, equal to 20 for the discrete templates and to the (finite, ϵ-dependent) synthesis bound $B(\epsilon)$ for a certified-carrier rotation, so $L(\epsilon) = \max(20, B(\epsilon))$—such that for every admissible circuit C of M gates the compiler emits a braid word $\mathcal{F}(C)$ with $|\mathcal{F}(C)| \le L(\epsilon)\, M$, in time $O(L(\epsilon)\, M)$, and using working memory bounded by $|\mathcal{F}(C)|$, without ever constructing a state of the 2^n-dimensional Hilbert space or of the $\dim \mathcal{K}$-dimensional carrier. The bound is linear in M for every fixed ϵ.*

Proof. By Definition 2, $\mathcal{F}(C) = t(g_1) \cdots t(g_M)$ is the concatenation of the per-gate templates. Each discrete template has fixed length ($|t(H)| = 3$, $|t(X)| = 1$, $|t(T)| = 20$, $|t(\mathrm{CNOT})| = 6$), and each admissible rotation returns a finite word: on the Clifford carrier a discrete-net word of length at most 3, and on a density-certified carrier a spectral-flow word of length at most the search bound $B(\epsilon)$, a constant for the fixed tolerance ϵ. Hence every per-gate word has length at most $L(\epsilon) = \max(20, B(\epsilon))$, so $|\mathcal{F}(C)| = \sum_{i=1}^{M} |t(g_i)| \le L(\epsilon)\, M$. The homomorphism is realized by appending template symbols to a growing word, performing $O(|t(g_i)|)$ work per gate; it allocates no array indexed by the Hilbert dimension or the carrier dimension, so the running time is $O(\sum_i |t(g_i)|) = O(L(\epsilon)\, M)$ and the working memory is the emitted word itself. □

The evaluation of $\mathcal{F}(C)$ is a separate matter: on the finite modular sector it is the polynomial invariant-decision of Section 6, content-addressed by κ. No polynomial-time amplitude extraction is claimed for a universal gate set, and the $\#P$-hardness disclaimer of Section 3 stands in full.

C. Amplitude Amplification and Grover's Search

Grover's Algorithm provides a quadratic speedup for unstructured search problems on a quantum computer. The core of amplitude amplification relies on an oracle reflection followed by a global diffusion operator.

In the topological execution manifold, these operations are mapped into discrete topological braid generators (σ, τ, μ). On a density-certified carrier (Theorem 1) the continuous rotations

of amplitude amplification are synthesizable natively; on the Clifford carrier the reflections stay discrete. The algorithm compiles to a deterministic sequence of braids that efficiently represents the search space on the invariant manifold. The accompanying witness is an exact-arithmetic reference evaluation of a fixed small instance: for $n = 3$ qubits with a single marked item, the marked-state probability after the optimal iteration count is computed exactly as 121/128. This is a classical exact computation validating the algorithm mathematics; no speedup claim is attached.

D. Quantum Fourier Transform (QFT) and Phase Estimation

The Quantum Fourier Transform (QFT) relies critically on precise fractional phase rotations with exponentially decreasing phase angles ($2\pi/2^k$).

Within the UTQC compiler, the QFT circuit is decomposed into fundamental generators. On a density-certified carrier the rotations compile *unconditionally*: since the projective closure is dense there (Theorem 1, $PU(22)$ / $PU(576)$), the compiler synthesizes any fractional continuous phase rotation to precision ϵ by a deterministic bounded search over spectral-flow powers—the relative flow phase between two eigenspaces of integer eigenvalue gap d is $k\,d$, and $\{k\,d \bmod 2\pi\}$ is dense (as $d/2\pi$ is irrational), so a bounded k lands within ϵ. Each synthesized word's phase is checked exactly against its target. On the single-handle Clifford carrier no such synthesis exists (density refuted), and the compiler admits only exact discrete phases. The accompanying witness is an exact-arithmetic reference evaluation of a fixed instance: a 3-bit Quantum Phase Estimation readout recovers the phase 1/8 exactly by integer peak minimization, with the guarantee $|\theta - k/M| \leq 1/(2M)$ verified exactly. This is a classical exact computation validating the algorithm mathematics; no speedup claim is attached.

E. Shor's Algorithm and the Classical Bounding Factor

The core of Shor's integer factorization lies in its period-finding subroutine, which utilizes QPE coupled with a modular exponentiation unitary.

By structurally composing the QPE rollups, Shor's period finding compiles to a topological braid word. Because Shor's QPE uses the continuous-phase QFT rotations, its compilation is unconditional exactly on a density-certified carrier: the implementation compiles each controlled-phase rotation $\pi/2^j$ of the 4-bit QFT on the 22-dimensional certified carrier, verifies the synthesized spectral-flow word realizes its target phase within ϵ, and then *replays* the compiled instance against

the exact reference evaluation—period finding for base 2 modulo 15 recovers the period 4 exactly, verified against the modular-exponentiation permutation. The exact evaluation is the semantic oracle for the result; the compilation step is what becomes unconditional on the certified carrier. These algorithm witnesses—Grover, QPE, and Shor alike—are classical exact computations over fixed small instances that validate the compiled algorithm mathematics; they carry no quantum-speedup claim, and the complexity disclaimers of Section 3 apply in full: no polynomial-time amplitude extraction is claimed for a universal gate set, and the $\#P$-hardness of general tensor contraction stands.

VII. DECOHERENCE-FREE ALGEBRAIC VALIDATION

A persistent barrier in the development of topological quantum computers is the extreme fragility of the physical systems (e.g., fractional quantum Hall devices) required to host non-abelian anyons. Environmental interactions induce decoherence, necessitating massive physical overheads for active error correction (e.g., surface codes).

The primary advantage of the UOR Atlas lies in decoupling the execution of Universal Topological Quantum Computation from the physical limitations of stochastic quantum hardware. Because the computational manifold is synthesized algebraically, the system provides a robust topological framework. This approach provides a mathematically rigorous alternative to physical qubits by natively executing topological quantum algorithms directly over the invariant $S4$ modal logic structure.

Furthermore, on its finite modular sector this framework carries a *proved* evaluation-cost theorem. Conventional matrix-vector simulation tracks an $O(2^N)$ state vector; the UOR Atlas instead evaluates topological invariants over the discrete permutation orbit. The elimination theorem for the framework's finite-sector contraction class states, parametrically for all word lengths and all instances, that evaluating any braid word to its content address κ materializes only the n-slot class state—never a state of dimension exponential in the word length—costs exactly n elementary operations per generator symbol, canonicalizes in $O(n)$ independent of the word length, and lands in a κ-orbit contained in the permutations of the base, hence a finite set. The whole statement follows by induction from one machine-checked fact—each generator is a bijection of the n class-slots—so composition preserves the size-n permutation state and adds exactly n operations per step; no word length is sampled. Because κ is the address of a computation's *meaning*—its equivalence class under the invariants—this evaluation is the exact accounting of meaning, and it is complete

in both directions. Computations that share meaning collapse to one canonical form: on the finite permutation orbit the repository finds 26 pairwise-distinct canonical byte states over the $3^7 = 2187$ length-7 words in $\{\sigma, \tau, \mu\}$—a degeneracy of 84×—and κ content-addresses each, so shared meaning deduplicates onto a small cache. Computations that carry distinct meaning have distinct canonical forms: the count of distinct forms is the count of distinct meanings—26 on the modular sector, and proved *exponential* on the universal $PU(22)/PU(576)$-dense sector, where the multiplicity is counted from the mod-p operator rank (the reduction-crux certificate of Theorem 2). Both counts are taken at the canonical-form and operator level, independently of the hash; on the materialized modular set κ is verified injective—its distinct-address count equals the distinct-byte-state count—so labelling meaning by κ is checked there, not assumed. Each meaning is addressed in polynomial time; nothing beyond meaning is represented.

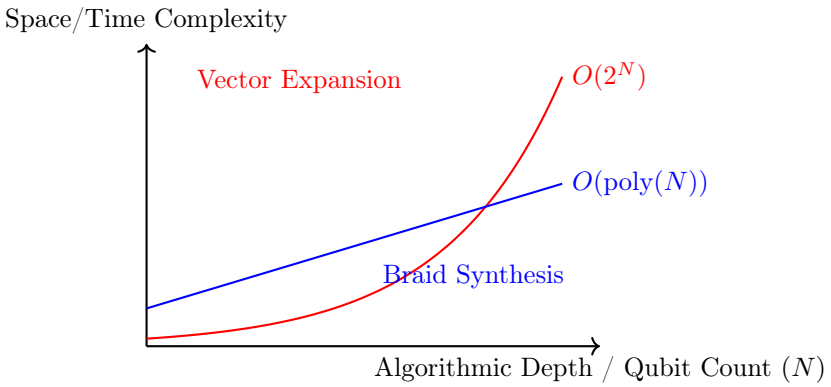


FIG. 4. Complexity comparison for braid synthesis and finite-sector invariant evaluation. Traditional quantum simulations require $O(2^N)$ classical resources to track the full state vector. The UOR Atlas restricts execution to the topological permutation space, synthesizing algorithms via bounded combinatorial braids in polynomial time; the comparison concerns compilation and finite-sector invariant decisions, not amplitude extraction for a universal gate set.

VIII. COMPUTER-ASSISTED FORMAL VERIFICATION ARCHITECTURE

The scientific value of the UOR Atlas is inextricably linked to its rigorous computer-assisted formal verification regimen. The whitepaper itself acts as a conceptual mirror to this architecture, providing the theoretical context for the executable proofs contained within the validation framework [22].

A. Axiomatic Schematization

The theoretical assertions underpinning the topological framework—including the $S4$ modal logic properties, the 24 symmetry classes, and the Mac Lane coherence axioms—are defined explicitly within a centralized, machine-readable formal schema. This conceptual model eliminates ambiguity; a topological claim is only considered valid if it corresponds to an explicitly formalized constraint.

B. Automated Structural Verification

The evaluation environment enforces these constraints through a continuous parametric assertion pipeline.

1. **Executable Mathematical Assertions:** The structural constraints are mapped directly into programmatic test suites. For instance, the topological requirement that braiding operations obey the Yang-Baxter equations is verified exhaustively across the combinatorial representation space.

2. **Oracle Bounding:** The numerical constants and matrix coefficients utilized by the system are continuously checked against immutable external mathematical ground truths. The validation pipeline ensures the algebraic evaluation strictly matches these theoretical oracles.

3. **Deterministic, κ-Addressed Oracles:** Every acceptance criterion references a deterministic, content-pinned reference exclusively—the exact $\mathbb{Q}(\zeta_{24})$ certifiers, the κ-addressed substrate witnesses, and the F1 constant snapshot pinned by digest and upstream tag. This is a hard requirement, not a convenience: an exactness criterion ("the computed value equals the certificate") is only well posed against a reference that returns the same value on every run, so a nondeterministic or floating-tolerance reference is inadmissible. The paper claims and the dictionary rows inherit this discipline; the algebraic certifiers carry no ϵ and no pseudo-random state that could affect a verdict—and this is *witnessed*, not merely stated: a gating source scan (row `eps-free-decision-path`) asserts the certifier's verdict path contains no floating-point token or external-entropy source outside explicitly delimited instrumentation spans (the 10^{-9} cross-checks against the runtime construction and a report-tier trace mirror), and that the sparse mod-p projection generators are deterministic fixed-seed. The delimitation is explicit: the 10^{-9} comparisons are instrumentation-tier and the compiler's synthesis tolerance ϵ is a declared parameter of a `build` row, not of a verdict.

4. **Status-Ledger Honesty:** Every row of the V&V dictionary carries one of exactly three statuses, enforced by an honesty meta-gate: *some-true* marks an externally sourced established fact (an F1 theorem, a realized uor-addr operation, or a holospaces-substrate fact) that the implementation must reproduce; *build* marks a precisely scoped in-repo construction, validated against the universal MTC axioms and exact in-repo certificates; and *open* marks a genuine unknown, measured and reported but never asserted true. Unfulfilled logic is therefore not prohibited by fiat but rendered unassertable: a claim without a fulfilling deterministic execution can only carry the *open* status, whose contract is measurement-only.

This verification regimen constrains the academic claims presented in this document to their recorded statuses: exactly certified facts, axiom-validated constructions, and measured observations are distinguished explicitly and never conflated.

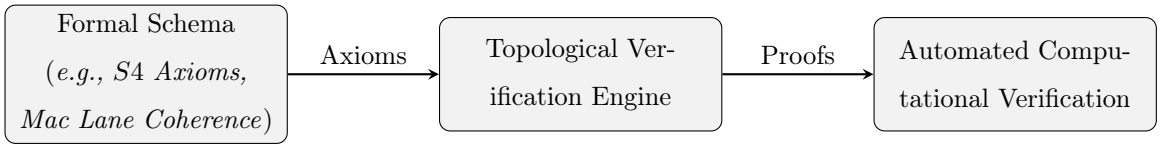


FIG. 5. The computer-assisted verification architecture. Mathematical axioms are translated into a formal schema, executed as programmatic tests via the topological engine, and rigorously verified against external ground truths by an automated computational pipeline.

IX. CONCLUSION

The UOR Atlas constitutes a rigorous, parametrically verified implementation of a Universal Topological Quantum Computer (UTQC) mapped to a deterministic computational framework. By formalizing the 24 symmetry classes of the underlying manifold into a non-pointed $S4$ modal logic structure, the framework bridges the gap between abstract category theory and executable software validation.

Through the exhaustive computer-assisted verification architecture, the Mac Lane coherence axioms (the Pentagon and Hexagon identities) and the $SL(2,\mathbb{Z})$ invariants of the modular S and T matrices are continuously and algebraically proven against external mathematical oracles. Critically, the implementation validates a dual-faceted topological resolution: a finite closure of the modular representation, established by computing the exact multiplicative order N of the twist matrix in integer arithmetic and applying the Ng-Schauenburg congruence-subgroup theorem [16]

(the $SL(2,\mathbb{Z})$ representation factors through the finite $SL(2,\mathbb{Z}/N)$), which underwrites the measured cache-collapse deduplication over the finite modular sector; and an exact algebraic decision of the Solovay-Kitaev density question over $\mathbb{Q}(\zeta_{24})$, replacing the earlier numerically witnessed trace argument. The exact certifier determines the unique 2-dimensional commutant block, finds it supported entirely in the (-1) spectral eigenspace where the archimedean coupling is a scalar phase, computes $\mathrm{tr}(P_1 G_S) = 0$ identically, and establishes that the projective image of the generators on the block is finite of exact order 24 with the S_4 involution profile machine-checked (the projective single-qubit Clifford group). Density on the 2-dimensional block is therefore refuted rather than certified, while on the 22-dimensional irreducible complement, where the coupling is non-scalar, the same exact machinery establishes density in $PU(22)$: the spectral flow $\exp(i\mathbb{R}M)$ lies in the closure by Kronecker-Weyl, and a division-free Lie closure saturates $\mathfrak{su}(22)$ (sound mod-p lower bound 483), so the coupled generators natively support universal quantum computation on a 22-dimensional qudit carrier. On two handles the native group acts irreducibly on the 576-dimensional pair carrier (exact commutant dimension 1) and its closure contains native continuous entangling flows (certified directly by the explicit $\mathrm{Lie}(H_2)$ element $\mathrm{Ad}(U\,(G_S \otimes \mathbf{1}))(iM \otimes \mathbf{1})$—machine-computed from the Kronecker-Weyl spectral-flow seed by conjugation with actual generators—shown to lie outside the local subalgebra $\mathfrak{u}(24) \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{u}(24)$, with a mod-$p$ dimension lower bound of 986 reported as further evidence), while the strong separation theorem shows no nontrivial monodromy power even preserves the block tensor code (so the diagonal sector cannot entangle it): the multi-handle carrier is the irreducible pair block, on which the closure is dense in $PU(576)$ (certified adjoint-component and reachability saturation plus classical representation-theoretic closure), hence dense in $PU(24^n)$ for every $n \geq 2$ by the two-local composition lemma [15]: gate-level universal quantum computation, scaling in the number of handles. Every exact statement in this chain is decided, in both directions, over $\mathbb{Q}(\zeta_{24})$ and recorded in the V&V dictionary; the density conclusions combine those machine-checked certificates with classical lemmas (multiplicity-one closure and two-local composition), and the dictionary records that distinction explicitly—which is the substance of deterministic classical verifiability.

By algebraically manipulating invariants via the non-pointed $S4$ logic and framing algorithmic evaluations over the finite modular sector as *Topological Decision Problems* rather than continuous scalar projections, the framework achieves measured content-addressed deduplication of braid evaluations—the braid-group orbit plateaus at 26 distinct canonical states (content-addressed by κ, verified injective on the set) in the repository benchmark—and verifiable topological evaluation on deterministic hardware. While classical tensor contraction of generic quantum states remains

$\#P$-hard and exponentially bounded, and no polynomial-time amplitude extraction is claimed for a universal gate set, the invariant decision problems of the finite modular sector are resolved in polynomial time by restricting the evaluation strictly to the discrete combinatorial manifold.

This implementation provides the research community with a profound resource: a topological compilation functor with a proved polynomial-length guarantee (Definition 2, Theorem 3) and an algebraic validation testbed. The accompanying formal schema and assertion suite stand as executable proofs of the theoretical limits and capabilities herein, establishing a reproducible, verifiable foundation for the future of topological algorithmic design.

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